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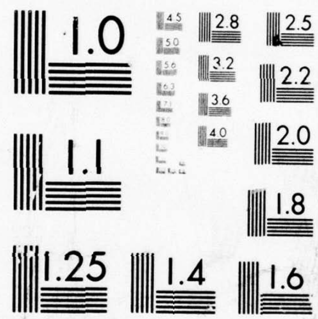


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# FOREIGN TECHNOLOGY DIVISION

ONE PROBLEM OF B. V. NEDENKO

by

D. G. Meyzler

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# UNEDITED MACHINE TRANSLATION

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ONE PROBLEM OF B. V. NEDENKO

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PREPARED BY:

TRANSLATION DIVISION
FOREIGN TECHNOLOGY DIVISION
WP-AFB, OHIO.

## U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

| Block | Italic | Transliteration | Block | Italic | Transliteration |
|---|---|---|---|---|---|
| А а | *А а* | A, a | Р р | *Р р* | R, r |
| Б б | *Б б* | B, b | С с | *С с* | S, s |
| В в | *В в* | V, v | Т т | *Т т* | T, t |
| Г г | *Г г* | G, g | У у | *У у* | U, u |
| Д д | *Д д* | D, d | Ф ф | *Ф ф* | F, f |
| Е е | *Е е* | Ye, ye; E, e* | Х х | *Х х* | Kh, kh |
| Ж ж | *Ж ж* | Zh, zh | Ц ц | *Ц ц* | Ts, ts |
| З з | *З з* | Z, z | Ч ч | *Ч ч* | Ch, ch |
| И и | *И и* | I, i | Ш ш | *Ш ш* | Sh, sh |
| Й й | *Й й* | Y, y | Щ щ | *Щ щ* | Shch, shch |
| К к | *К к* | K, k | Ъ ъ | *Ъ ъ* | " |
| Л л | *Л л* | L, l | Ы ы | *Ы ы* | Y, y |
| М м | *М м* | M, m | Ь ь | *Ь ь* | ' |
| Н н | *Н н* | N, n | Э э | *Э э* | E, e |
| О о | *О о* | O, o | Ю ю | *Ю ю* | Yu, yu |
| П п | *П п* | P, p | Я я | *Я я* | Ya, ya |

*ye initially, after vowels, and after ъ, ь; e elsewhere.
When written as ё in Russian, transliterate as yё or ё.

## RUSSIAN AND ENGLISH TRIGONOMETRIC FUNCTIONS

| Russian | English | Russian | English | Russian | English |
|---|---|---|---|---|---|
| sin | sin | sh | sinh | arc sh | $\sinh^{-1}$ |
| cos | cos | ch | cosh | arc ch | $\cosh^{-1}$ |
| tg | tan | th | tanh | arc th | $\tanh^{-1}$ |
| ctg | cot | cth | coth | arc cth | $\coth^{-1}$ |
| sec | sec | sch | sech | arc sch | $\operatorname{sech}^{-1}$ |
| cosec | csc | csch | csch | arc csch | $\operatorname{csch}^{-1}$ |

| Russian | English |
|---|---|
| rot | curl |
| lg | log |

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One Problem of B. V. Nedenko.

D. G. Meyzler.

1. On seminar on probability theory, for target/purpose of generalization of already known results [1], of B. V. Nedenkos was placed problem: to find class of maximum laws of maximum term $\eta_n$ first n from sequence of independent differently distributed random variables $\xi_n$. This problem has direct interest for statistics, namely - for that case when experimental conditions vary and the distribution function of the result of the i observation depends on i. Target/purpose of this work - to give, under some sufficiently general conditions, the solution recently formulated problem.

Let us note that the obtained by us results are found in close communication/connection with P. Levis's results, obtained by it in response to the question of A. Ya. Khinchin concerning maximum laws

for the standardized/normalized sums (see [2], §6).

Let us consider the sequence of the independent random quantities

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots$$

of subordinate with respect to the laws of the distribution

$$F_1(x), F_2(x), \ldots, F_n(x) \ldots$$

Let us assume

$$\eta_n = \max(\xi_1, \xi_2, \ldots, \xi_n) \qquad n=1, 2, \ldots$$

The distribution function of maximum term $\eta_n$ is

$$\Phi_n(x) = P\{\eta_n \leqslant x\} = F_1(x) \cdot F_2(x) \ldots F_n(x).$$

We let us say, that the law of distribution $\Phi(x)$ belongs to class G, if there is this sequence of distribution function $F_i(x)$ and such positive numbers $a_n$, that

$$\Phi(x) = \lim_{n\to\infty} \sum_{i=1}^{n} F_i(a_n x) \qquad (1)$$

and evenly relative to $i \; [1 \leqslant i \leqslant n]$

$$\lim_{n\to\infty} F_i(a_n x) = 1 \qquad (2)$$

for all x for which $\Phi(x) > 0$.

Our problem lies in the fact that, finding the of characteristic sign/criterion of the laws of class G.

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For convenience in the formulation of the proven sign/criteria, let us introduce the following designations:

Let $G^+$ it designates the set of those laws $\Phi(x)$ of class G for which $\Phi(0)=0$, and $G^-$ - respectively the set of those for which $\Phi(0)=1$. It is obvious, classes $G^+$ and $G^-$ do not intersect, and, as subsequently will be shown (lemma 4), occurs the equality

$$G^+ + G^- = G.$$

2. In the present work are demonstrated following theorems:

Theorem 1. For the law of distribution $\Phi(x)$ would belonging to class $G^+$, it is necessary and sufficient so that for any $\alpha$ $(0 < \alpha < 1)$ there would be nondecreasing function $\varphi_\alpha(x)$, such so that with all took place the equality

$$\Phi(x) = \Phi\left(\frac{x}{\alpha}\right) \cdot \varphi_\alpha(x). \qquad (3)$$

Theorem 2. For the law of distribution $\Phi(x)$ would belonging to class $G^-$, it is necessary and sufficient so that for any $\alpha$ $(0 < \alpha < 1)$ there would be nondecreasing function $\varphi_\alpha(x)$, such so that with all took place the equality

$$\Phi(x) = \Phi(\alpha x) \cdot \varphi_\alpha(x) \qquad (4)$$

and so that the law of distribution $\Phi(x)$ was continuous at point x = 0.

Theorem 3. The laws of the distribution $\Phi(x)$ of class G can have the quite larger one point of discontinuity, and, if $x_0$ there is a point of discontinuity, then

$$\text{или}^{(1)} \quad x_0=0 \text{ и}^{(4)} \ \Phi(x)=\varepsilon(x)=\begin{cases}0 & \text{при}^{(2)}\ x\leqslant 0,\\ 1 & \text{при}^{(2)}\ x>0,\end{cases}$$

$$\text{или}^{(1)} \quad x_0\neq 0,\ \Phi(x_0)=0 \text{ и}^{(4)}\ \Phi(x)>0 \text{ для}^{(3)}\ x>x_0.$$

Key: (1). or. (2). with. (3). for. (4). and.

Theorem 4. The laws of the distribution $\Phi(x)$ of class G do not have intervals of constancy besides those in which they are converted in 0 or 1.

Theorem 5. If for the sequence of the laws of distribution $\Phi_n(x)$ class G

$$\lim_{n\to\infty}\Phi_n(x)=\Phi(x), \tag{5}$$

where $\Phi(x)$ is a law of distribution, then $\Phi(x)$ belongs also to class G, and, if $\Phi(x)$ it belongs to sub-class $G^+$ (respectively $G^-$),

then, beginning with certain n, everything $\Phi_n(x)$ belong to sub-class $G^+$ (respectively $G^-$). In other words, class G, just as each of the sub-classes $G^+$ and $G^-$, they will close relative to transition to limit.

Theorem 6. If the law of distribution $\Phi(x)$ belongs to class $G^+$ (respectively $G^-$), then the law of the distribution

$$\Phi'(x) = \Phi(ax-b)$$

also belongs to class $G^+$ (respectively $G^-$), where a and b any positive numbers (moreover b satisfies condition $\Phi(-b)=1$).

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For the formulation of one additional property of the laws of class G, is necessary the following observation. Let F(x) any law of distribution, continuous in point x = 0 and such, that F(x) = 0 for x < 0, then

$$\bar{F}(x)=\begin{cases} F\left(-\frac{1}{x}\right) & \overset{(1)}{\text{для}}\ x<0, \\ 1 & \overset{(1)}{\text{для}}\ x\geqslant 0, \end{cases} \qquad (6)$$

Key: (1). for.

there is also the law of distribution, continuous in point x = 0 *. Conversely, if F (x) is the law of distribution, continuous in point x=0 and is such that $\bar{F}$ (x)=1 for x > 0, then

$$\bar{F}(x)=\begin{cases} 0 & \text{for } x\leqslant 0, \\ F\left(-\frac{1}{x}\right) & \text{for } x>0, \end{cases} \qquad (6')$$

is also a law of distribution, continuous in the point x=0*)

FOOTNOTE*. $\bar{F}$ (x) can and not be the law of distribution during the presence of discontinuity/interruption in point x=0. For example

for a single law we have:

$$\varepsilon(x)=\begin{cases}0 \text{ для } x\leqslant 0,\\ 1 \text{ для } x>0,\end{cases}\qquad \bar{\varepsilon}(x)=1 \text{ для всех } x.$$

Key: (a). for. (b). for all x/ ENDFOOTNOTE.

It is easy to note that $\bar{\bar{F}}(x) = F(x)$ and, therefore, transformations (5) and (6') establish/install one-to-one conformity between many of distribution function $F(x)$, continuous in point $x = 0$, which satisfy condition $F(x) = 0$ for $x < 0$ and, correspondingly, $\bar{F}(x)$, that satisfy condition $\bar{F}(x) = 1$ for $x > 0$.

Let us agree to call distribution functions $F(x)$ and $\bar{F}(x)$ corresponding.

Theorem 7. If the law of distribution $\Phi(x)$ $(\Phi(x) \neq \varepsilon(x))$ belongs to class $G^+$, then corresponding to it law $\bar{\Phi}(x)$ belongs to class $G^-$. Conversely, each law of distribution, corresponding for certain law of class $G^-$, belongs to class $G^+$.

3. Proof of theorems will be based on B. V. Nedenko's following lemma [1], which we will give without proof:

lemma 1. Let $F_n(x)$ and $\Phi(x)$ will be distribution functions, moreover $\Phi(x)$ is its own law. If with some sequences of real numbers

$a_n > 0,\ b_n,\ \alpha_n > 0,\ \beta_n$ they occur of the equality:

$$\text{(I)} \quad \lim_{n\to\infty} F_n(a_n x + b_n) = \Phi(x)$$

and

$$\text{(II)} \quad \lim_{n\to\infty} F_n(\alpha_n x + \beta_n) = \Phi(x),$$

then

$$\text{(III)} \quad \lim_{n\to\infty} \frac{a_n}{\alpha_n} = 1 \text{ и } \lim_{n\to\infty} \frac{b_n - \beta_n}{a_n} = 0.$$

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Conversely, if for the sequence of distribution function $F_n(x)$ during certain selection of real constants $a_n > 0$ and $b_n$ for all x occurs equality (I), where $\Phi(x)$ is certain nondecreasing function, then for any two sequences of the real numbers $\alpha_n > 0$ and $\beta_n$, which satisfy conditions (III), for all x also (II).

In order subsequently not to interrupt the presentation of the proofs of fundamental theorems, let us demonstrate preliminarily a series of the simple propositions:

lemma 2 (being the light/lung transformation of B. V. Nedenko's lemma 3 [1]). If with some $a_n > 0$ it occurs (1) and (2), moreover the law of distribution $\Phi(x)$ is its own, then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1. \qquad (7)$$

Proof. according to the condition

$$\Phi(x)=\lim_{n\to\infty}\prod_{i=1}^{n+1}F_i(a_{n+1}x)=\lim_{n\to\infty}\prod_{i=1}^{n}F_i(a_{n+1}x)\cdot F_{n+1}(a_{n+1}x),$$

and since for all x for which $\Phi(x)>0$,

$$\lim_{n\to\infty}F_n(a_nx)=1,\ \text{то}\ \ \Phi(x)=\lim_{n\to\infty}\prod_{i=1}^{n}F_i(a_{n+1}x).$$

This relationship/ratio, together with (1), on the basis of lemma 1, it gives (7).

Lemma 3. If with some $a_n>0$ they occur (1) and (2), moreover the law of distribution $\Phi(x)$ is its own, then or

$$\lim_{n\to\infty}a_n=\infty,\ \text{или}\ \lim_{n\to\infty}a_n=0.$$

Proof. Let us first of all note that there is no such subsequence $\{n_k\}$, for which

$$\lim_{k\to\infty}a_{n_k}=a$$

with 0 < a < ∞. After allowing contrary, on basis of lemma 1, we would have

$$\lim_{k\to\infty}\prod_{i=1}^{n_k}F_i(ax)=\Phi(x).$$

Since $\Phi(x)$ its own law of distribution, there is such ξ, that

$$0<\Phi(\xi)=\lim_{k\to\infty}\prod_{i=1}^{n_k}F_i(a\xi)<1.$$

But in view (2)

$$F_i(a\xi) = 1 \quad \text{for } i=1,2,\ldots,n,$$

whence

$$\Phi(\xi) = \lim_{k\to\infty} \sum_{i=1}^{n_k} F_i(a\xi) = 1,$$

which is led to contradiction.

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From another side it is not difficult to show that if for certain sequence of numbers $a_n>0$ it occurs (7), then many limit points of this sequence or consist of one (its own or improper) limit point, or everywhere dense (in the finite or infinite interval).

This observation, together with that which was demonstrated, shows the validity of our lemma.

Lemma 4. If with some $a_n>0$ they occur (1) and (2), moreover the law of distribution $\Phi(x)$ is its own, then either

$$\Phi(0) = 0 \text{ and } \lim_{n\to\infty} a_n = \infty$$

or

$$\Phi(0) = 1 \text{ and } \lim_{n\to\infty} a_n = 0.$$

Proof. Let us first of all note that it cannot be

$$0<\Phi(0)<1.$$

Actually, with x = 0 factors $F_i(a_n x)$ cease to depend on n and in view (2) we would have

$$F_i(0)=1 \quad \text{for} \quad i=1,2,\ldots,n,$$

whence

$$\Phi(0)=\lim_{n\to\infty}\prod_{i=1}^{n}F_i(0)=1,$$

which is led to contradiction. In view of that which was demonstrated and lemma 3, it suffices to show the incongruence of the relationship/ratios

$$\Phi(0)=0 \quad \text{and} \quad \lim_{n\to\infty}a_n=0 \tag{8}$$

and respectively

$$\Phi(0)=1 \quad \text{and} \quad \lim_{n\to\infty}a_n=\infty. \tag{9}$$

Let us assume that they occur (1), (2) and (8) and let ξ be such, that $0<\Phi(\xi)<1$. In view (8) we have ξ > 0. Let x > 0 be any number. Beginning from certain n it will be

$$a_n\xi < x \text{ and } F_i(a_n\xi) \leqslant F_i(x) \quad \text{for } i=1, 2, \ldots, n,$$

but in view (2)

$$\lim_{n\to\infty} F_i(a_n\xi) = 1 \quad \text{for } i=1, 2, \ldots, n,$$

whence

$$F_i(x) = 1 \quad \text{for } i=1, 2, \ldots, n.$$

Since the latter occurs with any x > 0, with any the natural i and $n$ $(i \leqslant n)$ $F_i(a_n\xi) = 1$, whence $\Phi(\xi)=1$, which is brought k contradictory.

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Analogously is proven the second part of the lemma. After allowing (9), we would have for any x < 0

$$\lim_{n\to\infty} F_i(a_n x) = 0 \quad \text{for } i=1, 2, \ldots, n,$$

whence $\Phi(x)=0$ for any x < 0 and, in view $\Phi(0)=1$, law $\Phi(x)$ it would be improper.

Lemma 5. If the law of distribution $\Phi(x)$ possesses those by

property, that with certain $\alpha(0 < \alpha < 1)$ there is nondecreasing function $\psi_\alpha(x)$ such, that in all points of the continuity of law $\Phi\left(\frac{x}{\alpha}\right)$ (respectively $\Phi(\alpha x)$) occurs the equality

$$\Phi(x) = \Phi\left(\frac{x}{\alpha}\right) \cdot \psi_\alpha(x), \qquad (10)$$

(respectively that

$$\Phi(x) = \Phi(\alpha x) \cdot \psi_\alpha(x)), \qquad (11$$

$$\Phi(x) = 0 \text{ for } x \leqslant 0.$$

(respectively $\Phi(x) = 1$ for $x > 0$).

Proof. Let us assume that this is erroneous. Since $\Phi(x)$ is continuous to the left, there is such $\xi < 0$, that

$$\Phi(\xi) > \Phi\left(\frac{\xi}{\alpha}\right) \geqslant 0,$$

moreover $\Phi(\alpha)$ is continuous at point $\frac{\xi}{\alpha}$. After allowing $\Phi\left(\frac{\xi}{\alpha}\right) = 0$, we obtain from (10) $\Phi(\xi) = 0$, which is impossible. After allowing $\Phi\left(\frac{\xi}{\alpha}\right) > 0$, we will obtain $\psi_\alpha(\xi) > 1$, that it is also impossible, since $\lim_{x\to\infty} \psi_\alpha(x) = 1$ and $\psi_\alpha(x)$ nondecreasing function.

The second part of the lemma is proven analogously.

Lemma 6. If the law of distribution $\Phi(x)$ possesses property (10) [with respect to (11) ] and there are such x(x >-∞), that $\Phi(x) = 0$, then among all possible $\psi_\alpha(x)$, that satisfy (10) [with respect to (11) ],

there is listribution function $\varphi_a(x)$ such, that $\Phi(x)$ and $\varphi_a(x)$ simultaneously they are converted into zero.

Proof. From (10), in view of continuity $\Phi(x)$ to the left and on the basis of lemma 5 it follows that there is such $x_0$, that $\Phi(x_0)=0$ and $\Phi(x)>0$ for $x>x_0$. Let us place

$$\varphi_a(x)=\begin{cases} 0 & \overset{(1)}{\text{для}}\ x\leqslant x_0, \\ \dfrac{\Phi(x)}{\Phi\left(\dfrac{x}{a}\right)} & \overset{(1)}{\text{для}}\ x>x_0. \end{cases}$$

Key: (1). for.

It is obvious, so that which was determined $\varphi_a(x)$ satisfies the condition of lemma.

The second part of the lemma is proven analogously.

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Let us note that if $\Phi(x)>0$ for all x, then such $\varphi_a(x)$ can not exist. It is easy to check that the law of the distribution

$$\Phi(x)=\begin{cases} -\dfrac{1}{x} & \overset{(1)}{\text{для}}\ x\leqslant -1, \\ 1 & \overset{(1)}{\text{для}}\ x>-1, \end{cases}$$

Key: (1). for.

satisfies condition (11) with any $\alpha(0<\alpha<1)$, that which is

determining unambiguously $\psi_a(x)$ taking the form

$$\psi_a(x)=\begin{cases} a & \overset{(1)}{\text{для}}\ x\leqslant-\frac{1}{a}, \\ -\frac{1}{x} & \overset{(1)}{\text{для}}\ -\frac{1}{a}\leqslant x\leqslant-1 \\ 1 & \overset{(1)}{\text{для}}\ x>-1. \end{cases}$$

Key: (1). for.

It is obvious, $\psi_a(x)$ not there is distribution function.

The lemma 7. If law of distribution $\Phi(x)$ possesses property (10) [with respect (to 11) ], then among all possible $\psi_a(x)$, that satisfy (10), [with respect (to 11) ] exists such, which (10) [with respect (to 11) ] occurs for all x.

Proof. On the basis of lemma 6, we can be bounded to the examination of those values x, for which $\Phi(x)>0$. Let $x_1$ and $x_2$ ($x_2 > x_1$) any two numbers ($\Phi(x_1)>0$). It suffices to show that

$$\frac{\Phi(x_1)}{\Phi\left(\frac{x_1}{a}\right)}\leqslant\frac{\Phi(x_2)}{\Phi\left(\frac{x_2}{a}\right)}. \tag{12}$$

Actually, let us take any $\xi, \eta$ $(\xi<x_1<\eta<x_2)$ such, that $\frac{\xi}{a}$ and $\frac{\eta}{a}$ they are the points of the continuity of function $\Phi(x)$. In view (10) we will have

$$\frac{\Phi(\xi)}{\Phi\left(\frac{\xi}{\alpha}\right)} \leqslant \frac{\Phi(\eta)}{\Phi\left(\frac{\eta}{\alpha}\right)}.$$

Let now $\xi \to x_1$, $\eta \to x_2$. Since $\Phi(x)$ is continuous to the left, last/latter relationship/ratio to us will give (12). The proof of the second part of the lemma is carried out by the same way.

Lemma 8. If the law of distribution $\Phi(x)$ possesses those by property, that for certain sequence of numbers $\alpha_n (0 < \alpha_n < 1)$, striving k 1, there are nondecreasing functions $\psi_{\alpha_n}(x)$ such, that with any natural n and any x

$$\Phi(x) = \Phi\left(\frac{x}{\alpha_n}\right) \cdot \psi_{\alpha_n}(x), \qquad (13)$$

[respectively $\Phi(x) = \Phi(\alpha_n x) \cdot \psi_{\alpha_n}(x)$],then for any a(0 < a < 1) there exists this nondecreasing function $\varphi_a(x)$, that for all x

$$\Phi(x) = \Phi\left(\frac{x}{a}\right) \cdot \varphi_a(x), \qquad (14)$$

[respectively $\Phi(x) = \Phi(ax) \cdot \varphi_a(x)$].

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Proof. On the basis of lemma 6, we can be bounded to the examination of those values x, for which $\Phi(x) > 0$. From (13) it follows that for any $x_1$ and $x_2$ $(x_2 \geqslant x_1,\ \Phi(x_1) > 0)$ with any natural n

$$\frac{\Phi(x_1)}{\Phi\left(\frac{x_1}{\alpha_n}\right)} \leqslant \frac{\Phi(x_2)}{\Phi\left(\frac{x_2}{\alpha_n}\right)}$$

or

$$\frac{\Phi(x_1)}{\Phi(x_2)} \leqslant \frac{\Phi\left(\frac{x_1}{\alpha_n}\right)}{\Phi\left(\frac{x_2}{\alpha_n}\right)}.$$

Since $\frac{x_2}{\alpha_n} \geqslant \frac{x_1}{\alpha_n}$, substituting into last/latter inequality $\frac{x_1}{\alpha_n}$ and $\frac{x_2}{\alpha_n}$ instead of $x_1$ and, correspondingly, $x_2$ we we will obtain

$$\frac{\Phi\left(\frac{x_1}{\alpha_n}\right)}{\Phi\left(\frac{x_2}{\alpha_n}\right)} \leqslant \frac{\Phi\left(\frac{x_1}{\alpha_n^2}\right)}{\Phi\left(\frac{x_2}{\alpha_n^2}\right)},$$

that together with the previous inequality it will give

$$\frac{\Phi(x_1)}{\Phi(x_2)} \leqslant \frac{\Phi\left(\frac{x_1}{\alpha_n^2}\right)}{\Phi\left(\frac{x_2}{\alpha_n^2}\right)}.$$

Analogously discussing, we will obtain that for any $x_1$ and $x_2$ ($x_2 \geqslant x_1$, $\Phi(x_1) > 0$) and natural n and m

$$\frac{\Phi(x_1)}{\Phi(x_2)} \leqslant \frac{\Phi\left(\frac{x_1}{\alpha_n^m}\right)}{\Phi\left(\frac{x_2}{\alpha_n^m}\right)}. \tag{15}$$

On the other hand, it is not difficult to show that if for certain sequence of numbers $\alpha_n (0<\alpha_n<1)$ $\lim_{n\to\infty} \alpha_n = 1$, that for any $\alpha (0<\alpha<1)$ it is possible to find such natural numbers $m = m(\alpha, n)$, ~~we~~ that $\lim_{n\to\infty} \alpha_n^m = \alpha$.

On the basis of this observation from (15) we obtain, that with any $\alpha (0<\alpha<1)$ occurs the inequality

$$\frac{\Phi(x_1)}{\Phi(x_2)} \leqslant \frac{\Phi\left(\frac{x_1}{\alpha}\right)}{\Phi\left(\frac{x_2}{\alpha}\right)} \tag{16}$$

for all $x_1$ and $x_2\,(x_2 \geqslant x_1,\ \Phi(x_1) > 0)$, which are the points of the continuity of function $\Phi\left(\frac{x}{a}\right)$.

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On the basis of lemma 7, this will occur also for any $x_1$ and $x_2\,(x_2 \geqslant x_1,\ \Phi(x_1) > 0)$. Let us assume

$$\varphi_a(x) = \begin{cases} 0 & \text{for those } x \text{ for which } \Phi(x) = 0, \\ \dfrac{\Phi(x)}{\Phi\left(\frac{x}{a}\right)} & \text{for remaining } x. \end{cases}$$

In view (16), so that which was determined $\varphi_a(x)$ will, obviously, satisfy condition (14). Proof of the second part of the lemma is carried out by the same way.

4. Let us pass to proof of formulated previously theorems:

Proof of theorem 1.

Sufficiency. From (3), on the basis of lemma 5 it follows that there is such $x_0$ $(x_0 \geqslant 0)$, that $\Phi(x_0) = 0$ and for $x > x_0$ $\Phi(x) > 0$. On the basis of lemma 5

$$\varphi_a(x) = \begin{cases} 0 & \text{for } x \leqslant x_0, \\ \dfrac{\Phi(x)}{\Phi\left(\frac{x}{a}\right)} & \text{for } x > x_0, \end{cases}$$

will be function distribution function during any $\alpha (0 < \alpha < 1)$. Let us place a = i/i + 1, then with any natural i

$$F_i(x) = \varphi_{\frac{i}{i+1}}\left(\frac{x}{i+1}\right) = \begin{cases} 0 & \overset{(1)}{\text{для}} \ x \leqslant x_0(i+1), \\ \dfrac{\Phi\left(\frac{x}{i+1}\right)}{\Phi\left(\frac{x}{i}\right)} & \overset{(1)}{\text{для}} \ x > x_0(i+1), \end{cases}$$

Key: (1). for.

it will be also distribution function. After assuming $a_n = n+1$, we will have

$$F_i(a_n x) = \varphi_{\frac{i}{i+1}}\left(\frac{a_n x}{i+1}\right) = \begin{cases} 0 & \overset{(1)}{\text{для}} \ x \leqslant \dfrac{x_0(i+1)}{n+1} \\ \dfrac{\Phi\left(\frac{n+1}{i+1}x\right)}{\Phi\left(\frac{n+1}{i}x\right)} & \overset{(1)}{\text{для}} \ x > \dfrac{x_0(i+1)}{n+1}. \end{cases} \quad (17)$$

Key: (1). for.

Is easy to check that for any $x > x_0$ and any natural $i$ $(1 \leqslant i \leqslant n)$ $\Phi\left(\frac{n+1}{i}x\right) \neq 0$ and $F_i(a_n x) \neq 0$.

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Let us assume

$$\Phi_n(a_n x) = \prod_{i=1}^{n} F_i(a_n x),$$

then for all $x > x_0$ $\Phi_n(a_n x) = \frac{\Phi(x)}{\Phi(a_n x)}$

and, in view $\lim_{n\to\infty} a_n = \infty$, will be $\lim_{n\to\infty} \Phi_n(a_n x) = \Phi(x)$

for all x for which $\Phi(x)>0$,

On the other hand, from (17) it follows that $F_n(a_n x) = 0$ for $x \leqslant x_0$, therefore, for all x it occurs (1). Finally, from (17) is obtained immediately (2), therefore, $\Phi(x)$ belongs to class G, and since $x_0 \geqslant 0$, $\Phi(0) = 0$ and $\Phi(x)$ belongs to class $G^+$.

Let us note that from lemmas 7 and 8 it follows that for equipment with class $G^+$ it suffices to require so that relationship/ratio (3) would be fulfilled for certain sequence $a_n (0 < a_n < 1)$, approaching 1, moreover it is sufficient so that this would occur at the points of continuity $\Phi(x)$.

Need. Let us assume that the law of distribution $\Phi(x)$ belongs to class $G^+$ this means that is made (1), (2) and $\Phi(0)=0$. Since each improper law of distribution $\Phi(x)$, which belongs to class $G^+$ (i.e. such, which $\Phi(0)=0$), satisfies condition (3), we can to assume that $\Phi(x)$ - its own law of distribution. On the basis of lemmas 2 and 4

$$\lim_{n\to\infty} a_n = \infty \quad \text{и} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

a therefore we can substitute into conformity to each index n this another index $m=m(\alpha, n)$, that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+m}} = \alpha, \tag{18}$$

where $\alpha$ - any number of interval (0.1). Let us assume

$$\Phi_n(x) = \prod_{i=1}^{n} F_i(a_n x),$$

then

$$\Phi_{n+m}(x) = \prod_{i=1}^{n+m} F_i(a_{n+m}x) = \prod_{i=1}^{n} F_i\left(\frac{a_{n+m}}{a_n} \cdot a_n x\right) \cdot \prod_{i=n+1}^{n+m} F_i(a_{n+m}x).$$

Let us designate

$$\varphi_{n,m}(x) = \prod_{i=n+1}^{n+m} F_i(a_{n+m}x),$$

then last/latter relationship/ratio is rewritten in the form

$$\Phi_{n+m}(x) = \Phi_n\left(\frac{a_{n+m}}{a_n} x\right) \cdot \varphi_{n,m}(x). \tag{19}$$

On the basis of the first Helly theorem, there is this subsequence $\varphi_{n_s, m_s}(x)$, that $\lim_{s\to\infty}\varphi_{n_s, m_s}(x)=\varphi_a(x)$, ~~rme~~ that $\varphi_a(x)$- certain nondecreasing (limited) function. From (19), in view (1), (18) and lemmas 1 we obtain, that in all points of the continuity of the law $\Phi\left(\frac{x}{a}\right)$

$$\Phi(x)=\Phi\left(\frac{x}{a}\right)\cdot\varphi_a(x)$$

with any $a\ (0<a<1)$. On the basis of lemma 7, last/latter relationship/ratio will occur with any x, which proves need (3).

Let us note that on the basis of lemma 6 it is necessary that relationship/ratio (3) would occur, also, when $\varphi_a(x)$, as the being distribution function.

Proof of theorem 2.

Sufficiency. Let us allow first, that $\Phi(x)>0$ for all x, then, in view (4),

$$\varphi_a(x)=\begin{cases} 0 & \overset{(1)}{\text{для}}\ x\leqslant\frac{a}{a-1}, \\ \frac{\Phi(x)}{\Phi(ax)} & \overset{(1)}{\text{для}}\ x>\frac{a}{a-1}, \end{cases}$$

Key: (1). for.

will be distribution function during any a (0 < α < 1). Let us assume then with any natural i

$a=\frac{i}{i+1}$

$$F_i(x)=\varphi_{\frac{i}{i+1}}[(i+1)x]=\begin{cases}0 & \text{для } x\leqslant -i,\\ \frac{\Phi[(i+1)x]}{\Phi(ix)} & \text{для } x>-i,\end{cases}$$

Key: (1). for.

it will be also distribution function. After assuming $a_n=\frac{1}{n+1}$ we will obtain

$$F_i(a_nx)=\varphi_{\frac{i}{i+1}}[(i+1)a_nx]=\begin{cases}0 & \overset{(1)}{\text{для}}\ x\leqslant -i(n+1),\\ \frac{\Phi\left(\frac{i+1}{n+1}x\right)}{\Phi\left(\frac{ix}{n+1}\right)} & \overset{(1)}{\text{для}}\ x>-i(n+1).\end{cases} \quad (20)$$

Key: (1). for.

Let us assume

$$\Phi_n(a_nx)=\prod_{i=1}^{n}F_i(a_nx).$$

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Since for any x and any i (1 ⩽ i ⩽ n), beginning with certain n, it

will be x > -i(n + 1), beginning with certain n,

$$F_i(a_n x) = \frac{\Phi\left(\frac{i+1}{n+1}x\right)}{\Phi\left(\frac{ix}{n+1}\right)} \text{ for all } i\ (1 \leqslant i \leqslant n) \text{ and } \Phi_n(a_n x) = \frac{\Phi(x)}{\Phi(a_n x)},$$

whence, on the basis of continuity condition in zero and of lemma 5, in view $\lim_{n\to\infty} a_n = 0$, we we obtain (1). From (20) it is obtained immediately (1), and since, in view of continuity condition in zero and lemma 5, $\Phi(0) = 1$, $\Phi(x)$ belongs to class $G^-$.

Let us allow now, which exists such $x_0$, that $\Phi(x_0) = 0$ and $\Phi(x) > 0$ for $x > x_0$. In view of lemma 5, $x_0 < 0$ and on the basis of lemma 6

$$\varphi_\alpha(x) = \begin{cases} 0 & \overset{(1)}{\text{для}}\ x \leqslant x_0, \\ \dfrac{\Phi(x)}{\Phi(\alpha x)} & \overset{(1)}{\text{для}}\ x > x_0, \end{cases}$$

Key: (1). for.

will be distribution function during any $\alpha$ $(0 < \alpha < 1)$. To analogously previous, after assuming $\alpha = \frac{i}{i+1}$, $a_n = \frac{1}{n+1}$, we we obtain, that

$$F_i(a_n x) = \varphi_{\frac{i}{i+1}}[(i+1)a_n x] = \begin{cases} 0 & \overset{(1)}{\text{для}}\ x \leqslant \dfrac{x_0(n+1)}{i+1}, \\ \dfrac{\Phi\left(\frac{i+1}{n+1}x\right)}{\Phi\left(\frac{i}{n+1}x\right)} & \overset{(1)}{\text{для}}\ x > \dfrac{x_0(n+1)}{i+1}, \end{cases} \tag{21}$$

Key: (1). for.

it will be distribution function, also, for any $x > x_0$ and any natural $i\ (1 \leqslant i \leqslant n)$ $\Phi\left(\frac{ix}{n+1}\right) \neq 0$ and $F_i(a_n x) \neq 0$. Let us assume

$$\Phi_n(a_n x) = \prod_{i=1}^{n} F_i(a_n x),$$

then, it is easy to check that for all $x > x_0$

$$\Phi_n(a_n x) = \frac{\Phi(x)}{\Phi(a_n x)}$$

and, in view $\lim_{n\to\infty} a_n = 0$, on the basis of continuity condition in zero and of lemma 5, will be made for all $x > x_0$ (1).

On the other hand, from (21) it follows that $F_n(a_n x) = 0$ for $x \leqslant x_0$, it consequently for all x occurs (1). From (21) it is obtained directly (2), which together on condition of continuity in zero, on the basis of lemma 5, proves the sufficiency of sign/criterion, also, in this case.

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Need. Let us assume that distribution function $\Phi(x)$ belongs to class $G^-$ i.e., they occur (1), (2) and $\Phi(0)=1$. Since each improper law of distribution $\Phi(x)$, which belongs to class $G^-$ (i.e. such, which $\Phi(0) = 1$), satisfies the conditions of theorem, we can to assume that $\Phi(x)$

its own law of distribution. On the basis of lemmas 2 and 4

$$\lim_{n\to\infty} a_n = 0 \text{ and } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1,$$

a therefore we can place in conformity to each index n this another index $m=m(\alpha, n)$, that

$$\lim_{n\to\infty}\frac{a_{n+m}}{a_n} = \alpha, \quad (22)$$

where $\alpha$ is any number of interval (0.1). To analogously previous, after assuming

$$\Phi_n(x) = \prod_{i=1}^{n} F_i(a_n x),$$

we we come to relationship/ratio (19) and in view (22) we obtain (4). Finally from (4) and condition $\Phi(0)=1$, on the basis of lemma 5 we obtain the need for the continuity of law $\Phi(x)$ for point x = 0.

The observation, done by us in the proof of theorem 1 relative to the possibility of the weakening of sufficient conditions, is applicable, obviously, and to theorem 2, but for the laws of the distribution $\Phi(x)$ of class $G^-$ which are converted into zero at end point, it is possible to somewhat amplify the necessary condition, since in this case it is necessary that relationship/ratio (4) would occur, also, when $\varphi_\alpha(x)$, as the being distribution function.

Proof of theorem 3.

Since the laws of class $G^-$, according to theorem 2, are continuous in point x = 0, first assertion of theorem sufficient to demonstrate on the assumption that $\Phi(x)$ belongs the kl. to class $G^+$ Actually, if $\Phi(x)$ class $G^+$ it had a discontinuity/interruption at point x = 0, then there would be such an a > 0, which for any x > 0 would be

$$\Phi(x) > a > 0,$$

whence, in view (3), we would have

$$\lim_{x\to+0} \varphi_a(x) = \lim_{x\to+0} \frac{\Phi(x)}{\Phi\left(\frac{x}{a}\right)} = 1,$$

whence $\Phi(x) = \Phi\left(\frac{x}{a}\right)$ for x > 0, and then since $\Phi(0) = 0$, то $\Phi(x) = \varepsilon(x)$.

Let us allow now, that the second assertion of theorem not is correct and let $x_0$ will be the point of the discontinuity of law $\Phi(x)$, moreover $\Phi(x_0) > 0$.

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Let us allow first, which $\Phi(x)$ belongs to class $G^+$, consequently $x_0 > 0$ there is such an a > 0, that for any $x > x_0$

$$\Phi(x_0) + a < \Phi(x). \qquad (23)$$

Since $\Phi(x)$ is continuous to the left, will be located such $\xi(0 < \xi < x_0)$ that

$$\frac{\Phi(\xi)}{\Phi(x_0)} > 1 - a \tag{24}$$

and with any $a (0 < a < 1)$ it will be

$$\frac{\Phi(x_0)}{\Phi\left(\frac{x_0}{a}\right)} \geqslant \frac{\Phi(\xi)}{\Phi\left(\frac{\xi}{a}\right)}.$$

Specifically, when $a = \frac{\xi}{x_0}$ we obtain

$$\frac{\Phi(x_0)}{\Phi\left(\frac{x_0^2}{\xi}\right)} \geqslant \frac{\Phi(\xi)}{\Phi(x_0)}$$

and in view (24)

$$\frac{\Phi(x_0)}{\Phi\left(\frac{x_0^2}{\xi}\right)} > 1 - a. \tag{25}$$

However, since $\xi < x_0$, that $\frac{x_0^2}{\xi} > x_0$ and in view (23) will be

$$\Phi(x_0) + a < \Phi\left(\frac{x_0^2}{\xi}\right)$$

or

$$\frac{\Phi(x_0)}{\Phi\left(\frac{x_0^2}{\xi}\right)} < 1 - \frac{a}{\Phi\left(\frac{x_0^2}{\xi}\right)} \leqslant 1 - a,$$

which contradicts (25).

In perfect analogy is proven the case, when $\Phi(x)$ belongs to class $G^-$ taking into account that $x_0 < 0$, since for $x \geqslant 0$ $\Phi(x)$ is knowingly continuous.

Let us note the following interesting fact: whatever was the law $\Phi(x)$ of class G, the law

$$F(x)=\begin{cases}0 & \text{для } x\leqslant a,\\ \Phi(x) & \text{для } x>a,\end{cases}$$

Key: (1). for.

where a (any real number) also belongs to class G. This observation gives, obviously, rule for the construction disruptive laws of class G.

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Proof of theorem 4.

Let $\Phi(x)$ will be the law of class G and let us assume that there are such $\xi_1$ and $\xi_2$ $(\xi_2>\xi_1)$, that

$$0<\Phi(\xi_1)=\Phi(\xi_2)<1. \qquad (26)$$

Since $\Phi(x)$ belongs to class G, from this it follows that $\xi_1\cdot\xi_2>0$.

Let us allow first, which $\xi_2>\xi_1>0$; then $\Phi(x)$ belongs to class $G^+$ and occurs (3) with any $\alpha(0<\alpha<1)$. After assuming in (3) $\alpha=\frac{\xi_1}{\xi_2}$, $x=\xi_1$, we

will obtain

$$\Phi(\xi_1) = \Phi(\xi_2)\,\varphi_a(\xi_1)$$

and in view (26) $\varphi_a(\xi_1)=1$. Since $\varphi_a(x)$ nondecreasing function, a $\frac{\xi_2}{\alpha} > \xi_1$, also $\varphi_a\left(\frac{\xi_1}{\alpha}\right)=1$. Substituting in (3) $x=\xi_2=\frac{\xi_1}{\alpha}$, we will obtain

$$\Phi(\xi_2) = \Phi\left(\frac{\xi_1}{\alpha^2}\right)\cdot\varphi_a\left(\frac{\xi_1}{\alpha}\right),$$

whence

$$\Phi(\xi_1) = \Phi\left(\frac{\xi_1}{\alpha}\right) = \Phi\left(\frac{\xi_1}{\alpha^2}\right).$$

Discussing analogously, we come to the equality

$$\Phi(\xi_1) = \Phi\left(\frac{\xi_1}{\alpha^n}\right),$$

where n is any natural number. But $\lim\limits_{n\to\infty}\frac{\xi_1}{\alpha^n}=\infty$, consequently

$$\lim_{n\to\infty}\Phi\left(\frac{\xi_1}{\alpha^n}\right)=1,$$

whence $\Phi(\xi_1)=1$, which contradicts (26).

After assuming that $\xi_1<\xi_2<0$ and $\Phi(x)$ belongs to class $G$, we, discussing analogously, let us arrive at the relation

$$\Phi(\xi_1) = \Phi(\alpha^n\xi_1),$$

where n is any natural number. Since $\lim\limits_{n\to\infty}\alpha^n\xi_1 = 0$, $\Phi(0)=1$ and $\Phi(x)$ is continuous at point x = 0, $\Phi(\xi_1)=1$, which also contradicts (26).

Proof of theorem 5.

Let us first of all show that if for the distribution functions $\Phi_n(x)$ of class $G^+$ it occurs (5), then $\Phi(x)$ there is also class $G^+$.

Actually, with any x and any $\alpha (0 < \alpha < 1)$

$$\Phi_n(x) = \Phi_n\left(\frac{x}{\alpha}\right) \varphi_\alpha^{(n)}(x) \quad \text{for } n = 1, 2, \ldots$$

where $\varphi_\alpha^{(n)}(x)$ - are some nondecreasing functions.

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On the basis of the first Helly theorem, it is possible to find this subsequence $\{n_k\}$, that

$$\lim_{n \to \infty} \varphi_\alpha^{(n_k)}(x) = \varphi_\alpha(x),$$

where $\varphi_\alpha(x)$ there is also certain nondecreasing function, moreover last/latter relationship/ratio will occur for all x, with any $\alpha (0 < \alpha < 1)$. Thus, in view (5) for all x with any $\alpha (0 < \alpha < 1)$ will be

$$\Phi(x) = \Phi\left(\frac{x}{\alpha}\right) \varphi_\alpha(x),$$

consequently, $\Phi(x)$ there is a class $G^+$.

Analogously it is proven, which if for the distribution functions of class $G^-$ occurs (5), then $\Phi(x)$ there is also class $G^-$.

Let now $\Phi_n(x)$ will be the sequence of the laws of the listribution of class G, let $\Phi_v^+(x)$ will be all those laws of the distribution of sequences $\Phi_n(x)$, which belong to class $G^+$ and respectively $\Phi_s^-(x) - G^-$. At least one of the sequences $\{v\}$ and $\{s\}$ it is infinite and, in view of recently that which was demonstrated, if $\{v\}$ [respectively $\{s\}$] it is infinite, then $\Phi(x)$ belongs to class $G^+$ (respectively $G^-$).

On the other hand, since classes $G^+$ and $G^-$, obviously, do not intersect, of (5) it follows that one of the sequences $\{v\}$ and $\{s\}$ it must be final, which, obviously, proves our theorem.

Proof of theorem 6.

Let $\Phi(x)$ will be the law of class $G^+$, and $x_0$ - the number, which satisfies the relationship/ratios: $\Phi(x_0)=0$ and $\Phi(x)>0$ for $x > x_0$.

Let us show first of all, that the law of the distribution

$$\Phi'(x) = \Phi(x-b),$$

where b is any positive number, also belongs to class $G^+$. For this, it suffices to show that for any x and $y (x \geqslant y > x_0 + b)$ and with any $\alpha (0 < \alpha < 1)$

$$\frac{\Phi'(x)}{\Phi'\left(\frac{x}{\alpha}\right)} \geqslant \frac{\Phi'(y)}{\Phi'\left(\frac{y}{\alpha}\right)}$$

or

$$\frac{\Phi(x-b)}{\Phi\left(\frac{x}{a}-b\right)} \geqslant \frac{\Phi(y-b)}{\Phi\left(\frac{y}{a}-b\right)}.$$

Let us assume that with some $x_1 > y_1 > x_0 + b$ and certain $\alpha_1$ $(0 < \alpha_1 < 1)$ is fulfilled the inequality

$$\frac{\Phi(x_1-b)}{\Phi\left(\frac{x_1}{\alpha_1}-b\right)} < \frac{\Phi(y_1-b)}{\Phi\left(\frac{y_1}{\alpha_1}-b\right)}. \tag{27}$$

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Since $\Phi(x)$ belongs to class $G^+$, with any $\alpha\,(0<\alpha<1)$

$$\frac{\Phi(x_1-b)}{\Phi\left(\frac{x_1-b}{\alpha}\right)} \geqslant \frac{\Phi(y_1-b)}{\Phi\left(\frac{y_1-b}{\alpha}\right)};$$

specifically, when $\alpha = \frac{\alpha_1 x_1 - \alpha_1 b}{x_1 - \alpha_1 b}$ will be

$$\frac{\Phi(x_1-b)}{\Phi\left(\frac{x_1}{\alpha_1}-b\right)} \geqslant \frac{\Phi(y_1-b)}{\Phi\left[\frac{(y_1-b)(x_1-\alpha_1 b)}{\alpha_1(x_1-b)}\right]} \tag{28}$$

(since with so determined $\alpha$ it occurs equality $\frac{x_1-b}{\alpha} = \frac{x_1}{\alpha_1}-b$). It is easy to check that for $b > 0$, $x_1 - y_1 > 0$, $0 < \alpha_1 < 1$ will be

$$\frac{y_1}{\alpha_1} - b > \frac{(y_1-b)(x_1-\alpha_1 b)}{\alpha_1(x_1-b)},$$

in view of which from (28) we obtain

$$\frac{\Phi(x_1-b)}{\Phi\left(\frac{x_1}{\alpha_1}-b\right)} \geqslant \frac{\Phi(y_1-b)}{\Phi\left(\frac{y_1}{\alpha_1}-b\right)},$$

which contradicts (27).

On the other hand, if $\Phi(x)$ belongs to class $G^+$ then, obviously, $\Phi(ax)$ also belongs to class $G^+$ with any a > 0, consequently the law of distribution $\Phi'(x)=\Phi(ax-b)$ such belongs to class $G^+$ so forth.

The proof of the second part of the theorem for the laws of class $G^-$, is carried out by the same way; condition $\Phi(-b)=1$ ensures the continuity of law $\Phi'(x)=\Phi(ax-b)$ at point x = 0.

Let us note that with b of negative this theorem, generally speaking, is inaccurate, as is evident from the following examples:

$$F_1(x)=\begin{cases} 0 & \overset{(1)}{\text{для}}\ x\leqslant 1,\\ x^2-1 & \overset{(1)}{\text{для}}\ 1<x\leqslant\sqrt{2},\\ 1 & \overset{(1)}{\text{для}}\ x>\sqrt{2},\end{cases}\qquad F_2(x)=\begin{cases} -\frac{1}{x} & \overset{(1)}{\text{для}}\ x\leqslant -1,\\ 1 & \overset{(1)}{\text{для}}\ x>-1.\end{cases}$$

Key: (1). for.

It is not difficult to check that $F_1(x)$ (with respect $F_2(x)$) belongs to class $G^+$ (respectively $G^-$), nevertheless law $F'_1(x)$ = $F_1(x-b)$ [respectively $F'_2(x)$ = $F_2(x-b)$] with any negative b to class $G^+$ (respectively $G^-$) does not belong.

The proof of theorem 7.

Let the law of distribution $\Phi(x)$ belong to class $G^+$, therefore, $\Phi(x) = 0$ for x ≤ 0 and with any $\alpha (0 < \alpha < 1)$ there is this nondecreasing function $\varphi_\alpha(x)$, which for all x occurs (3).

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Since according to condition $\Phi(x) \neq \varepsilon(x)$, according to theorem 3, $\Phi(x)$ it is continuous at point x = 0, whence

$$\overline{\Phi}(x) = \begin{cases} \Phi\left(-\frac{1}{x}\right) & \overset{(1)}{\text{для}}\ x < 0, \\ 1 & \overset{(1)}{\text{для}}\ x \geqslant 0, \end{cases}$$

Key: (1). for.

will be the law of distribution corresponding to law $\Phi(x)$. In order to show that $\overline{\Phi}(x)$ belongs to class $G^-$, let us note that if we place

$$\overline{\varphi}_\alpha(x) = \begin{cases} \varphi_\alpha\left(-\frac{1}{x}\right) & \overset{(1)}{\text{для}}\ x < 0, \\ 1 & \overset{(1)}{\text{для}}\ x \geqslant 0, \end{cases}$$

Key: (1). for.

that $\overline{\varphi}_\alpha(x)$ will be also the nondecreasing function, since according to lemma 6 we can to assume that $\varphi_\alpha(x)$ and $\Phi(x)$ simultaneously they are converted into zero.

But in view (3), with any x

$$\overline{\Phi}(x) = \overline{\Phi}(ax) \cdot \overline{\varphi}_a(x),$$

a since our transformation retains continuity in zero, $\overline{\Phi}(x)$ belongs to class $G$.

The second part of the theorem is proven analogously.

In conclusion I express deep appreciation to my leader B. V. Nedenko for the setting of problem and management/manual during its solution.

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